The Theory of Evolutes and Envelopes

From First Principles to Advanced Geometric and Conic Applications

A Rigorous Masterclass on Envelopes, Centers of Curvature, Caustics, and the Evolute of an Ellipse

Contents

Introduction and Geometric Motivation . . . 2
Theory of Envelopes of a One-Parameter Family of Curves . . . 2
Definition of a Family and its Envelope . . . 2
The Dual Rigorous Proofs for the Envelope Condition . . . 2
Proof 1: Limiting Intersection of Consecutive Curves . . . 3
Proof 2: Tangency Condition via the Multivariable Chain Rule . . . 3
Differential Geometry of Curvature and the Osculating Circle . . . 4
Curvature and Radius of Curvature . . . 4
The Center of Curvature and the Osculating Circle . . . 4
The Evolute of a Curve . . . 5
Formal Definition . . . 5
Proof of the Fundamental Equivalence Theorem . . . 5
Huygens' Involute Theorem . . . 5
Comprehensive Case Study: The Evolute of an Ellipse . . . 6
Method 1: Envelope of the Family of Normals . . . 6
Step 1: Partial Differentiation . . . 6
Step 2: Introducing the Common Ratio . . . 6
Step 3: Eliminating into the Normal Line . . . 7
Step 4: Applying the Pythagorean Identity . . . 7
Step 5: Final Elimination . . . 7
Method 2: Center of Curvature Derivation . . . 7
Evaluating the Abscissa α of the Center of Curvature: . . . 8
Evaluating the Ordinate β of the Center of Curvature: . . . 8
Locus Elimination: . . . 8
Anatomy, Geometry, and Caustic Bifurcation . . . 9
Cusps and the Eccentricity Threshold . . . 9
The Caustic Boundary for Real Normals . . . 10
Fully Worked Numerical Case Study . . . 11
Evolute Parameters and Cusps . . . 11
Testing an External Point . . . 11
Quartic Roots and Co-normal Coordinates . . . 11
Evolutes of Other Classical Curves . . . 12
The Parabola $y^2 = 4ax$ . . . 12
The Hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ . . . 12
The Cycloid . . . 12

Summary of Key Results 13

Introduction and Geometric Motivation

In elementary coordinate geometry, a line is typically investigated as a standalone entity: a tangent touches a curve at one point, and a normal stands perpendicular to that tangent.

However, when a curve generates an infinite **family** of lines—such as the set of all tangents or the set of all normals—an entirely new geometric object emerges:

1. The **envelope** of the family of lines is the boundary curve that touches every single member of the family.
2. The **evolute** of the original curve is defined as the locus of its centers of curvature.

A fundamental theorem discovered by Christiaan Huygens (1673) proves that these two seemingly distinct definitions are mathematically identical:

The Fundamental Equivalence Theorem:
The evolute of any regular plane curve is identical to the envelope of its normal lines.

In the study of the ellipse, this boundary is of paramount practical importance: it forms an astroid-like caustic curve that divides the plane into distinct domains. Points inside this boundary admit **four distinct real normals**, while points outside admit only **two real normals**.

—

Theory of Envelopes of a One-Parameter Family of Curves

Definition of a Family and its Envelope

Let $F(x, y, \theta) = 0$ represent an equation in Cartesian coordinates (x, y) that also depends on a single continuous real parameter $\theta \in I \subseteq \mathbb{R}$. For each fixed value of θ, $F(x, y, \theta) = 0$ defines a plane curve C_θ.

The collection $\{C_\theta\}_{\theta \in I}$ is called a **one-parameter family of curves**.

> **Definition (Envelope):** An **envelope** of a one-parameter family of curves $F(x, y, \theta) = 0$ is a curve E such that:
> 1. At each point of E, it is tangent to some member of the family.
> 2. Each member of the family is tangent to E at one or more points.

The Dual Rigorous Proofs for the Envelope Condition

To determine the envelope, standard analysis prescribes eliminating the parameter θ between:

$$F(x, y, \theta) = 0 \quad \text{and} \quad \frac{\partial F}{\partial \theta}(x, y, \theta) = 0$$

We now establish why this system of equations holds, via two complementary mathematical formulations.

Proof 1: Limiting Intersection of Consecutive Curves

Consider two distinct curves belonging to the family corresponding to two nearby parameter values:

$$C_\theta : F(x, y, \theta) = 0$$

$$C_{\theta+\Delta\theta} : F(x, y, \theta + \Delta\theta) = 0$$

Let (x, y) be an intersection point of these two curves. Since (x, y) satisfies both equations, it also satisfies any linear combination. Specifically, subtracting the first equation from the second and dividing by $\Delta\theta \neq 0$:

$$\frac{F(x, y, \theta + \Delta\theta) - F(x, y, \theta)}{\Delta\theta} = 0$$

We now examine what happens as the two curves become arbitrarily close, i.e., in the limit as $\Delta\theta \to 0$:

$$\lim_{\Delta\theta \to 0} \frac{F(x, y, \theta + \Delta\theta) - F(x, y, \theta)}{\Delta\theta} = \frac{\partial F}{\partial \theta}(x, y, \theta) = 0$$

Therefore, the limiting intersection point (x, y) of two consecutive members of the family must satisfy:

$$F(x, y, \theta) = 0 \quad \text{and} \quad \frac{\partial F}{\partial \theta}(x, y, \theta) = 0$$

Since the envelope is precisely the locus of these limiting intersection points of neighboring curves, eliminating θ between these two simultaneous equations eliminates the parameter and yields the implicit equation of the envelope in (x, y).

Proof 2: Tangency Condition via the Multivariable Chain Rule

Let the envelope E be parameterized by θ:

$$\boldsymbol{r}(\theta) = (x(\theta), y(\theta))$$

where each point $(x(\theta), y(\theta))$ is the point where the envelope touches the curve C_θ.

Since $(x(\theta), y(\theta))$ lies on C_θ, the identity holds:

$$F(x(\theta), y(\theta), \theta) = 0$$

Differentiating this identity with respect to the independent parameter θ using the multivariable chain rule:

$$\frac{dF}{d\theta} = \frac{\partial F}{\partial x}\frac{dx}{d\theta} + \frac{\partial F}{\partial y}\frac{dy}{d\theta} + \frac{\partial F}{\partial \theta} = 0$$

Now examine the geometric tangency condition:
1. The velocity (tangent) vector to the envelope is:

$$\boldsymbol{T} = \left(\frac{dx}{d\theta}, \frac{dy}{d\theta}\right)$$

2. The gradient vector to the curve $F(x, y, \theta) = 0$ is the normal vector:

$$\nabla F = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}\right)$$

Since the envelope is **tangent** to the curve C_θ, the tangent vector $\boldsymbol{T}$ of the envelope must be perpendicular to the normal vector ∇F of the curve:

$$\nabla F \cdot \boldsymbol{T} = \frac{\partial F}{\partial x}\frac{dx}{d\theta} + \frac{\partial F}{\partial y}\frac{dy}{d\theta} = 0$$

Substituting this orthogonality relation directly into the chain rule expansion:

$$0 + \frac{\partial F}{\partial \theta} = 0 \Rightarrow \frac{\partial F}{\partial \theta} = 0$$

This confirms that at every point of contact on the envelope, $\frac{\partial F}{\partial \theta}$ must vanish identically. Solving $F = 0$ and $\frac{\partial F}{\partial \theta} = 0$ isolates the envelope points.

—

Differential Geometry of Curvature and the Osculating Circle

Curvature and Radius of Curvature

Let C be a regular plane curve parameterized by arc length s. Let ψ be the angle that the tangent line makes with the positive x-axis.

Definition (Curvature): The **curvature** κ of a plane curve measures the rate of turning of the tangent vector per unit distance traveled along the curve:

$$\kappa = \frac{d\psi}{ds}$$

The **radius of curvature** ρ is the reciprocal of the curvature:

$$\rho = \frac{1}{|\kappa|}$$

For a curve defined in Cartesian coordinates by $y = f(x)$:

$$\tan\psi = y' = \frac{dy}{dx}$$

Differentiating with respect to x:

$$\sec^2\psi \frac{d\psi}{dx} = y'' \Rightarrow \left(1 + (y')^2\right)\frac{d\psi}{dx} = y''$$

Since $ds = \sqrt{1 + (y')^2}\,dx$:

$$\kappa = \frac{d\psi}{ds} = \frac{d\psi/dx}{ds/dx} = \frac{y''/\left(1 + (y')^2\right)}{\sqrt{1 + (y')^2}} = \frac{y''}{\left(1 + (y')^2\right)^{3/2}}$$

Thus, the radius of curvature is:

$$\rho = \frac{\left(1 + (y')^2\right)^{3/2}}{|y''|}$$

The Center of Curvature and the Osculating Circle

At any point $P(x, y)$ on the curve, the **circle of curvature** (or **osculating circle**) is the circle that shares the same position, the same first derivative (tangent), and the same second derivative (curvature) as the curve.

The center $C(\alpha, \beta)$ of this circle is called the **center of curvature**. Since C lies on the normal to the curve at a distance ρ along the inward normal unit vector:

$$\alpha = x - \frac{y'\left[1 + (y')^2\right]}{y''}$$

$$\beta = y + \frac{1 + (y')^2}{y''}$$

The Evolute of a Curve

Formal Definition

> **Definition (Evolute):** The **evolute** of a curve C is the locus of all its centers of curvature $C(\alpha, \beta)$ as the point P traverses C.

Proof of the Fundamental Equivalence Theorem

We now prove that the envelope of the family of normals to C is precisely the evolute of C.

Proof: Let the curve C be given by $y = f(x)$. The equation of the normal line at $P(x_0, y_0)$ with slope $m_N = -1/y_{0'}$ is:

$$y - y_0 = -\frac{1}{y_{0'}}(x - x_0) \Rightarrow (x - x_0) + y_{0'}(y - y_0) = 0$$

Treating x_0 as the parameter θ of the family of normals, we define:

$$N(x, y, x_0) = (x - x_0) + y_{0'}(x_0)[y - y_0(x_0)] = 0$$

To find the envelope of this family of normals, we differentiate N partially with respect to the parameter x_0:

$$\frac{\partial N}{\partial x_0} = -1 + y_{0''}[y - y_0] + y_{0'}[-y_{0'}] = -1 - (y_{0'})^2 + y_{0''}(y - y_0) = 0$$

Solving this equation for y:

$$y_{0''}(y - y_0) = 1 + (y_{0'})^2 \Rightarrow y = y_0 + \frac{1 + (y_{0'})^2}{y_{0''}} = \beta$$

Now substitute this expression for $(y - y_0)$ back into the normal equation $N = 0$:

$$(x - x_0) + y_{0'}\left[\frac{1 + (y_{0'})^2}{y_{0''}}\right] = 0 \Rightarrow x = x_0 - \frac{y_{0'}\left[1 + (y_{0'})^2\right]}{y_{0''}} = \alpha$$

These coordinates $(x, y) = (\alpha, \beta)$ are identical to the coordinates of the center of curvature.

Hence, the envelope of the normals is precisely the locus of the centers of curvature. This establishes the theorem. $\square$

Huygens' Involute Theorem

An immediate consequence of this equivalence is Huygens' Theorem:

"The length of an arc of the evolute between two centers of curvature C_1 and C_2 equals the difference in the radii of curvature of the original curve at the corresponding points P_1 and P_2, provided the curvature varies monotonically."

$$\Delta s_{\text{evolute}} = |\rho_2 - \rho_1|$$

If a flexible, inextensible string is wrapped tightly around the evolute and then unwound while held taut, its free endpoint traces out the original curve C (called the **involute**).

—

Comprehensive Case Study: The Evolute of an Ellipse

Consider the standard central ellipse:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (a > b > 0)$$

Let $c^2 = a^2 - b^2$. In parametric form, any point on the ellipse is:

$$P(\theta) = (a\cos\theta, b\sin\theta)$$

Method 1: Envelope of the Family of Normals

The equation of the normal to the ellipse at eccentric angle θ is:

$$\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2$$

Let $F(x, y, \theta) = \frac{ax}{\cos\theta} - \frac{by}{\sin\theta} - (a^2 - b^2) = 0$.

Step 1: Partial Differentiation

Differentiating F partially with respect to θ:

$$\frac{\partial F}{\partial \theta} = \frac{ax\sin\theta}{\cos^2\theta} + \frac{by\cos\theta}{\sin^2\theta} = 0$$

Equating both terms:

$$\frac{ax\sin\theta}{\cos^2\theta} = -\frac{by\cos\theta}{\sin^2\theta}$$

Dividing both sides by $\sin\theta\cos\theta$:

$$\frac{ax}{\cos^3\theta} = -\frac{by}{\sin^3\theta}$$

Step 2: Introducing the Common Ratio

Let this common ratio be λ:

$$\frac{ax}{\cos^3\theta} = -\frac{by}{\sin^3\theta} = \lambda$$

Solving for $\cos\theta$ and $\sin\theta$:

$$\cos\theta = \left(\frac{ax}{\lambda}\right)^{1/3}, \qquad \sin\theta = \left(-\frac{by}{\lambda}\right)^{1/3}$$

Step 3: Eliminating into the Normal Line

Substitute these into the normal line equation $F(x, y, \theta) = 0$:

$$\frac{ax}{\left(\frac{ax}{\lambda}\right)^{1/3}} - \frac{by}{\left(-\frac{by}{\lambda}\right)^{1/3}} = a^2 - b^2$$

$$(ax)^{2/3}\lambda^{1/3} + (by)^{2/3}\lambda^{1/3} = a^2 - b^2$$

$$\lambda^{1/3}\left[(ax)^{2/3} + (by)^{2/3}\right] = a^2 - b^2$$

Step 4: Applying the Pythagorean Identity

Using $\cos^2\theta + \sin^2\theta = 1$:

$$\left(\frac{ax}{\lambda}\right)^{2/3} + \left(-\frac{by}{\lambda}\right)^{2/3} = 1$$

$$\frac{(ax)^{2/3} + (by)^{2/3}}{\lambda^{2/3}} = 1 \Rightarrow \lambda^{2/3} = (ax)^{2/3} + (by)^{2/3}$$

Taking the square root of both sides:

$$\lambda^{1/3} = \left[(ax)^{2/3} + (by)^{2/3}\right]^{1/2}$$

Step 5: Final Elimination

Substitute this expression for $\lambda^{1/3}$ back into the result of Step 3:

$$\left[(ax)^{2/3} + (by)^{2/3}\right]^{1/2} \cdot \left[(ax)^{2/3} + (by)^{2/3}\right] = a^2 - b^2$$

$$\left[(ax)^{2/3} + (by)^{2/3}\right]^{3/2} = a^2 - b^2$$

Raising both sides to the power $2/3$:

$$\boxed{(ax)^{2/3} + (by)^{2/3} = \left(a^2 - b^2\right)^{2/3}}$$

This is the Cartesian equation of the evolute of the ellipse.

Method 2: Center of Curvature Derivation

We now derive the same result from the curvature perspective.

Parametrically:

$$x = a\cos\theta, \qquad y = b\sin\theta$$

$$\frac{dx}{d\theta} = -a\sin\theta, \qquad \frac{dy}{d\theta} = b\cos\theta$$

The first derivative is:

$$y' = \frac{dy/d\theta}{dx/d\theta} = \frac{b\cos\theta}{-a\sin\theta} = -\frac{b}{a}\cot\theta$$

The second derivative is:

$$y'' = \frac{d(y')}{d\theta} \cdot \frac{d\theta}{dx} = \left(\frac{b}{a} \csc^2 \theta \right) \cdot \left(-\frac{1}{a \sin \theta} \right) = -\frac{b}{a^2 \sin^3 \theta}$$

We calculate $1 + (y')^2$:

$$1 + (y')^2 = 1 + \frac{b^2 \cos^2 \theta}{a^2 \sin^2 \theta} = \frac{a^2 \sin^2 \theta + b^2 \cos^2 \theta}{a^2 \sin^2 \theta}$$

Evaluating the Abscissa α of the Center of Curvature:

$$\alpha = x - \frac{y' \left[1 + (y')^2 \right]}{y''} = a \cos \theta - \frac{\left(-\frac{b \cos \theta}{a \sin \theta} \right) \left(\frac{a^2 \sin^2 \theta + b^2 \cos^2 \theta}{a^2 \sin^2 \theta} \right)}{-\frac{b}{a^2 \sin^3 \theta}}$$

$$\alpha = a \cos \theta - \frac{\cos \theta (a^2 \sin^2 \theta + b^2 \cos^2 \theta)}{a} = \frac{\cos \theta [a^2 - a^2 \sin^2 \theta - b^2 \cos^2 \theta]}{a}$$

Since $a^2 - a^2 \sin^2 \theta = a^2 \cos^2 \theta$:

$$\alpha = \frac{\cos \theta (a^2 \cos^2 \theta - b^2 \cos^2 \theta)}{a} = \frac{a^2 - b^2}{a} \cos^3 \theta$$

Evaluating the Ordinate β of the Center of Curvature:

$$\beta = y + \frac{1 + (y')^2}{y''} = b \sin \theta + \frac{\frac{a^2 \sin^2 \theta + b^2 \cos^2 \theta}{a^2 \sin^2 \theta}}{-\frac{b}{a^2 \sin^3 \theta}}$$

$$\beta = b \sin \theta - \frac{\sin \theta (a^2 \sin^2 \theta + b^2 \cos^2 \theta)}{b} = \frac{\sin \theta [b^2 - a^2 \sin^2 \theta - b^2 \cos^2 \theta]}{b}$$

Since $b^2 - b^2 \cos^2 \theta = b^2 \sin^2 \theta$:

$$\beta = \frac{\sin \theta (b^2 \sin^2 \theta - a^2 \sin^2 \theta)}{b} = -\frac{a^2 - b^2}{b} \sin^3 \theta$$

Locus Elimination:

Thus, the parametric equations of the evolute are:

$$\alpha = \frac{a^2 - b^2}{a} \cos^3 \theta, \qquad \beta = -\frac{a^2 - b^2}{b} \sin^3 \theta$$

Isolating $\cos \theta$ and $\sin \theta$:

$$\cos \theta = \left(\frac{a\alpha}{a^2 - b^2} \right)^{1/3}, \qquad \sin \theta = \left(-\frac{b\beta}{a^2 - b^2} \right)^{1/3}$$

Squaring and adding:

$$\left(\frac{a\alpha}{a^2 - b^2} \right)^{2/3} + \left(-\frac{b\beta}{a^2 - b^2} \right)^{2/3} = \cos^2 \theta + \sin^2 \theta = 1$$

$$(a\alpha)^{2/3} + (b\beta)^{2/3} = (a^2 - b^2)^{2/3}$$

Replacing (α, β) with current coordinates (x, y), we recover the identical equation. $\square$

—

Anatomy, Geometry, and Caustic Bifurcation

Cusps and the Eccentricity Threshold

The evolute is an astroid-like curve with four sharp cusps:

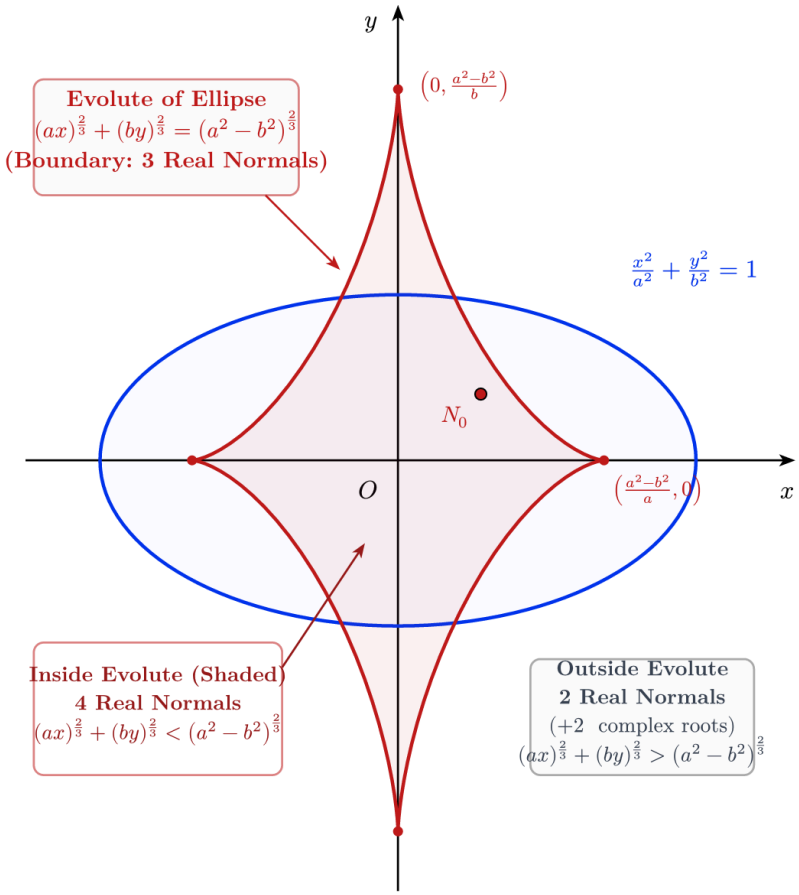
1. **Major Axis Cusps:** When $\theta = 0, \pi$, we obtain $\beta = 0$ and:
$$\alpha = \pm\frac{a^2 - b^2}{a}$$
2. **Minor Axis Cusps:** When $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$, we obtain $\alpha = 0$ and:
$$\beta = \pm\frac{a^2 - b^2}{b}$$

Notice the striking distinction between the major and minor axis cusps:

- Since $\frac{a^2-b^2}{a} = a\left(1 - \frac{b^2}{a^2}\right) = ae^2 < a$, the horizontal cusps **always** lie strictly inside the ellipse.
- Along the minor axis, the cusp distance is $\frac{a^2-b^2}{b}$. Comparing this with the semi-minor axis b:
$$\frac{a^2 - b^2}{b} > b \Leftrightarrow a^2 - b^2 > b^2 \Leftrightarrow a^2 > 2b^2 \Leftrightarrow e > \frac{1}{\sqrt{2}} \approx 0.7071$$

Geometric Consequence of the Eccentricity Threshold:

1. If $e < \frac{1}{\sqrt{2}}$, the vertical cusps lie **inside** the ellipse. The entire evolute is contained inside the ellipse.
2. If $e = \frac{1}{\sqrt{2}}$, the vertical cusps land **precisely** on the minor axis vertices $(0, \pm b)$.
3. If $e > \frac{1}{\sqrt{2}}$, the vertical cusps poke **outside** the ellipse along the minor axis.



The Caustic Boundary for Real Normals

Recall that the eccentric angle θ of any normal passing through an external point (h, k) satisfies the quartic equation in $t = \tan\frac{\theta}{2}$:

$$bkt^4 + 2(ah + c^2)t^3 + 2(ah - c^2)t - bk = 0$$

The discriminant Δ of this polynomial vanishes **precisely on the evolute**.

Position of Point (h, k)	Algebraic Nature	Geometric Meaning
$(ah)^{2/3} + (bk)^{2/3} < (a^2 - b^2)^{2/3}$ (Inside Evolute)	4 distinct real roots	Exactly **4 real normals** can be drawn from (h, k) to the ellipse.
$(ah)^{2/3} + (bk)^{2/3} = (a^2 - b^2)^{2/3}$ (On Evolute)	Repeated real root (double root)	Exactly **3 real normals** can be drawn (two of which coincide).
$(ah)^{2/3} + (bk)^{2/3} > (a^2 - b^2)^{2/3}$ (Outside Evolute)	2 real roots, 2 complex conjugate roots	Only **2 real normals** can be drawn from (h, k) to the ellipse.

—

Fully Worked Numerical Case Study

Consider the ellipse with parameters:

$$a = 3.6, \qquad b = 2.0$$

$$c^2 = a^2 - b^2 = 3.6^2 - 2.0^2 = 12.96 - 4.0 = 8.96$$

$$e = \frac{\sqrt{8.96}}{3.6} \approx 0.8315 > \frac{1}{\sqrt{2}}$$

Evolute Parameters and Cusps

1. Major axis cusps: $\left(\pm\frac{c^2}{a}, 0\right) = \left(\pm\frac{8.96}{3.6}, 0\right) \approx (\pm 2.489, 0)$.
2. Minor axis cusps: $\left(0, \pm\frac{c^2}{b}\right) = \left(0, \pm\frac{8.96}{2.0}\right) = (0, \pm 4.480)$.
3. Critical boundary value:

$$\left(a^2 - b^2\right)^{2/3} = (8.96)^{2/3} \approx 4.3139$$

Testing an External Point

Let the test point be $N_0(h, k) = (1.0, 0.8)$.

$$(ah)^{2/3} + (bk)^{2/3} = (3.6 \times 1.0)^{2/3} + (2.0 \times 0.8)^{2/3} = 3.6^{2/3} + 1.6^{2/3}$$

$$3.6^{2/3} \approx 2.3489, \qquad 1.6^{2/3} \approx 1.3680$$

$$(ah)^{2/3} + (bk)^{2/3} \approx 2.3489 + 1.3680 = 3.7169$$

Comparing with the boundary threshold:

$$3.7169 < 4.3139 \Rightarrow (ah)^{2/3} + (bk)^{2/3} < \left(a^2 - b^2\right)^{2/3}$$

Thus, the point $N_0(1.0, 0.8)$ lies **strictly inside the evolute**. Exactly 4 real normals can be drawn!

Quartic Roots and Co-normal Coordinates

Substituting $(h, k) = (1.0, 0.8)$ into the quartic equation:

$$1.6t^4 + 2(3.6 + 8.96)t^3 + 2(3.6 - 8.96)t - 1.6 = 0$$

$$1.6t^4 + 25.12t^3 - 10.72t - 1.6 = 0$$

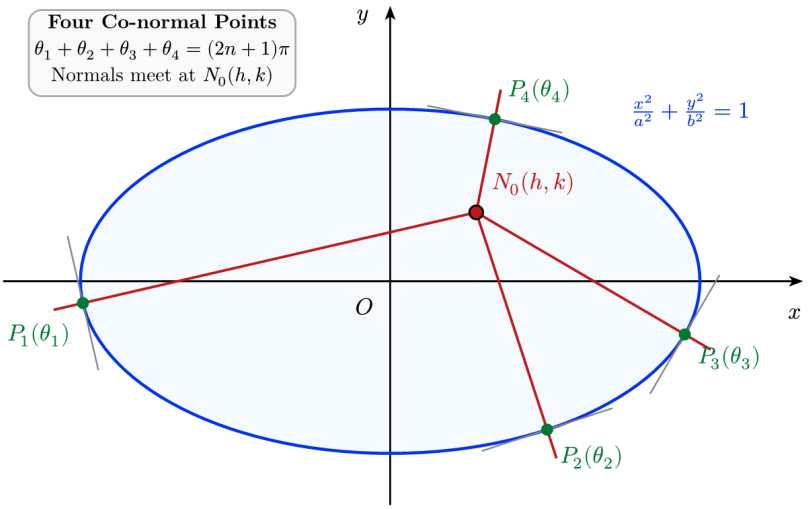
The four real roots are:

$$t_1 \approx -15.673 \Rightarrow \theta_1 \approx -172.70^\circ \Rightarrow P_1 = (-3.571, -0.254)$$

$$t_2 \approx -0.572 \Rightarrow \theta_2 \approx -59.55^\circ \Rightarrow P_2 = (1.824, -1.724)$$

$$t_3 \approx -0.158 \Rightarrow \theta_3 \approx -18.01^\circ \Rightarrow P_3 = (3.424, -0.618)$$

$$t_4 \approx 0.704 \Rightarrow \theta_4 \approx 70.26^\circ \Rightarrow P_4 = (1.216, 1.882)$$



Check:

$$\theta_1 + \theta_2 + \theta_3 + \theta_4 = -172.70^\circ - 59.55^\circ - 18.01^\circ + 70.26^\circ = -180.00^\circ = -\pi$$

This confirms the Co-normal Theorem ($\sum \theta_i = (2n+1)\pi$) with precision!

—

Evolutes of Other Classical Curves

The Parabola $y^2 = 4ax$

1. Normal line: $y = mx - 2am - am^3$.
2. Parameter m: $F(x, y, m) = m^3 a + m(2a - x) + y = 0$.
3. Differentiating with respect to m:

$$\frac{\partial F}{\partial m} = 3am^2 + (2a - x) = 0 \Rightarrow m^2 = \frac{x - 2a}{3a}$$

4. Eliminating m:

$$27ay^2 = 4(x - 2a)^3$$

This is the famous **semi-cubical parabola** with its cusp at $(2a, 0)$. From points inside the semi-cubical parabola ($27ay^2 < 4(x - 2a)^3$), three real normals can be drawn to the parabola.

The Hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Following the exact envelope procedure with $P(a\sec\theta, b\tan\theta)$:

$$(ax)^{2/3} - (by)^{2/3} = \left(a^2 + b^2\right)^{2/3}$$

The Cycloid

For a cycloid parameterized by:

$$x = a(t - \sin t), \qquad y = a(1 - \cos t)$$

The evolute is another **congruent cycloid** of identical radius a, displaced by π along the horizontal axis and shifted downwards by $2a$! This self-replicating geometric property enabled Huygens to design the isochronous cycloidal pendulum clock.

—

Summary of Key Results

Concept	Formula / Core Result
Envelope Condition	$F(x,y,\theta)=0$ and $\frac{\partial F}{\partial \theta}(x,y,\theta)=0$
Curvature of $y=f(x)$	$\kappa=\frac{y''}{\left(1+(y')^2\right)^{3/2}}$
Radius of Curvature	$\rho=\frac{1}{\lvert\kappa\rvert}=\frac{\left(1+(y')^2\right)^{3/2}}{\lvert y''\rvert}$
Center of Curvature	$\alpha=x-\frac{y'\left[1+(y')^2\right]}{y''},\quad \beta=y+\frac{1+(y')^2}{y''}$
Evolute Definition	Locus of centers of curvature ≡ Envelope of normal lines
Evolute of Ellipse	$(ax)^{2/3}+(by)^{2/3}=\left(a^2-b^2\right)^{2/3}$
Evolute Cusps (Ellipse)	$\left(\pm\frac{a^2-b^2}{a},0\right)$ and $\left(0,\pm\frac{a^2-b^2}{b}\right)$
Cusp Outward Condition	$e>\frac{1}{\sqrt{2}}$ (Minor axis cusps extend outside ellipse)
Normal Regions (Ellipse)	Inside: 4 real normals | On: 3 real normals | Outside: 2 real normals